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ON ϕ-CHAINED RINGS AND ϕ-PSEUDO-VALUATION RINGS

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ABSTRACT. Let R be a commutative ring with 1 such that $Nil(R)$ is a divided prime ideal of R. Then R is called a *ϕ-chained ring* if for every $x, y \in R \setminus Nil(R)$ either $x \mid y$ or $y \mid x$. Also, R is called a *ϕ-pseudo valuation ring* if for every $x, y \in R \setminus Nil(R)$ either $x \mid y$ or $y \mid xm$ for each nonunit $m \in R$. We show that a quasi-local ring R with maximal ideal M containing a nonzerodivisor of R is a ϕ-pseudo valuation ring iff $M : M$ is a ϕ-chained ring. We show that a ϕ-pseudo-valuation ring is a pullback of a ϕ-chained ring. Also, we show that for each $n \geq 1$ there is a ϕ-chained ring of Krull dimension n that is not a chained ring.

1. INTRODUCTION

We assume throughout that all rings are commutative with $1 \neq 0$. We begin by recalling some background material. As in [12], an integral domain R, with quotient field K, is called a *pseudo-valuation domain (PVD)* in case each prime ideal P of R is *strongly prime*, in the sense that $xy \in P, x \in K, y \in K$ implies that either $x \in P$ or $y \in P$. In [5], Anderson, Dobbs and the author generalized the study of pseudo-valuation domains to the context of arbitrary rings (possibly with nonzero zerodivisors). Recall from [5] that a prime ideal P of R is said to be *strongly prime (in R)* if aP and bR are comparable (under inclusion) for all $a, b \in R$. A ring R is called a *pseudo-valuation ring (PVR)* if each prime ideal of R is strongly prime. A PVR is necessarily quasilocal [5, Lemma 1(b)]; a chained ring is a PVR [[5], Corollary 4]; and an integral domain is a PVR if and only if it is a PVD (cf. [1, Proposition 3.1], [2, Proposition 4.2], and [6, Proposition 3]). Recall from [7] and [10] that a prime ideal P of R is called *divided* inclusion) to every ideal of R. A ring R is called a *divided ring* if every prime ideal of R is

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divided. In [8], the author gives another generalization of PVDs to the context of arbitrary rings (possibly with nonzero zerodivisors). Recall from [8] that for a ring R with total quotient ring $T(R)$ such that $Nil(R)$ is a divided prime ideal of R, let $\phi : T(R) \longrightarrow K := R_{Ni(R)}$ such that $\phi(a/b) = a/b$ for every $a \in R$ and every $b \in R \setminus Z(R)$. Then ϕ is a ring homomorphism from $T(R)$ into K, and ϕ restricted to R is also a ring homomorphism from R into K given by $\phi(x) = x/1$ for every $x \in R$. A prime ideal Q of $\phi(R)$ is called a *K-strongly prime* if $xy \in Q$, $x \in K, y \in K$ implies that either $x \in Q$ or $y \in Q$. If each prime ideal of $\phi(R)$ is K-strongly prime, then $\phi(R)$ is called a *K-pseudo-valuation ring (K-PVR)*. A prime ideal P of R is called a *ϕ-strongly prime* if $\phi(P)$ is a K-strongly prime ideal of $\phi(R)$. If each prime ideal of R is ϕ-strongly prime, then R is called a *ϕ-pseudo-valuation ring ($\phi - PVR$)*. It is shown in [8, Corollary 7(2)] that a ring R is a ϕ-PVR if and only if $Nil(R)$ is a divided prime ideal and for every $a, b \in R \setminus Nil(R)$, either $a \mid b$ in R or $b \mid ac$ in R for each nonunit $c \in R$. Also, it is shown in [9, Theorem 2.6] that for each $n \geq 0$ there is a ϕ-PVR of Krull dimension n that is not a PVR.

In this paper, we introduce the new concept:*ϕ-chained rings*. We show that a ϕ-chained ring is a ϕ-pseudo-valuation ring. We show that for each $n \geq 0$ there is a ϕ-chained ring of Krull dimension n that is not a chained ring. Among other results, we show that a ϕ-pseudo- valuation ring is a pullback of a ϕ-chained ring.

The following notation will be used throughout. Let R be a ring. Then $T(R)$ denotes the total quotient ring of R, Nil(R) denotes the set of nilpotent elements of R, $Z(R)$ denotes the set of zerodivisors of R, dim(R) denotes the Krull dimension of R, $Spec(R)$ denotes the set of all prime ideals of R, and if B is an R-module, then $Z(B)$ denotes the set of zerodivisors on B, that is, $Z(B) = \{x \in R : xy = 0$ in B for some $y \neq 0$ and $y \in B\}$. If I is an ideal of R, then $Rad(I)$ denotes the radical ideal of $I(inR)$. If $Nil(R)$ is a divided prime ideal of R, then K denotes the ring $R_{Nil(R)}$ and ϕ denotes the ring homomorphism from $T(R)$ into K given by $\phi(a/b) = a/b$ for each $a \in R$ and for each $b \in R \setminus Z(R)$.

Remark. Observe that $\phi(x) = x/1$ for each $x \in R$. Also, observe that by [8, Proposition 3], K is quasilocal ring with maximal ideal $Nil(\phi(R)) = \phi(Nil(R))$. Hence, each $x \in K \setminus Nil(\phi(R))$ is a unit of K.

We summarize some basic properties of PVRs and ϕ-PVRs in the following proposition.

Proposition 1.1. (1) *A PVR is a divided ring* [5, Lemma 1].

(2) *A ϕ-PVR is a divided ring* [8, Proposition 4].

(3) *An integral domain is a PVR iff it is a ϕ-PVR iff it is a PVD(* [1, Proposition 3.1], [2, Proposition 4.2], [6, Proposition 3], *and* [8]*).*
(4) *A ring R is a PVR if and only if for every $a, b \in R$, either $a \mid b$ or $b \mid ac$ for every nonunit c of R* [5, Theorem 5].
(5) *A ring R is a ϕ-PVR if and only if $Nil(R)$ is a divided prime ideal of R and for every $a, b \in R \setminus Nil(R)$, either $a \mid b$ in R or $b \mid ac$ in R for every nonunit $c \in R$.*
(6) *If R is a PVR or a ϕ-PVR, then $Nil(R)$ and $Z(R)$ are divided prime ideals of R (*[5], [8]*). Observe that if R is a ϕ-PVR, then $Nil(R)$ is a divided prime ideal of R by the definition.*

Our non-domain examples of ϕ-chained rings are provided by the idealization construction $R(+)B$ arising from a ring R and an R-module B as in Huckaba [13, Chapter VI]. We recall this construction. For a ring R, let B be an R-module. Consider $R(+)B = \{(r, b) : r \in R \text{ and } b \in B\}$, and let (r, b) and (s, c) be two elements of $R(+)B$. Define :

(1) $(r, b) = (s, c)$ if $r = s$ and $b = c$.
(2) $(r, b) + (s, c) = (r + s, b + c)$.
(3) $(r, b)(s, c) = (rs, bs + rc)$.

Under these definitions $R(+)B$ becomes a commutative ring with identity. In the following proposition, we state some basic properties of $R(+)B$.

Proposition 1.2. *Let R be a ring, B be an R-module, and $Z(B)$ be the set of zerodivisors on B. Then:*

(1) *The ideal J of $R(+)B$ is prime if and only if $J = P(+)B$ where P is a prime ideal of R. Hence, $dim(R) = dim(R(+)B)$* [13, Theorem 25.1].
(2) *$(r, b) \in Z(R(+)B)$ if and only if $r \in Z(R) \cup Z(B)$* [13, Theorem 25.3].
(3) *$(r, b) \in R(+)B$ is a unit of $R(+)B$ if and only if r is a unit of R* [13, Theorem 25.1].

2. ϕ-Chained rings

Throughout this section R denotes a ring with 1 such that $Nil(R)$ is a divided prime ideal of R. We start this section with the following definition.

Definition 1. . For a ring R, we say that $\phi(R)$ is a *K-chained ring (K-CR)* if for each $x \in K \setminus \phi(R)$, we have $x^{-1} \in \phi(R)$. If $\phi(R)$ is a K-CR , then we say that R is a *ϕ-chained ring (ϕ-CR).*

Remark. (1) Observe that every chained ring is a ϕ-chained ring.

(2) Observe that an integral domain is a valuation domain (chained ring) iff it is a ϕ-chained ring.
(3) Observe that K is a K-CR.

We recall the following result.

Lemma 2.1. [8, Proposition 3(3)]. *Let $x \in K$ and write $x = a/b$ for some $a \in R$ and for some $b \in R \setminus Nil(R)$. Then $x \in \phi(R)$ if and only if $b \mid a$ in R.*

Proposition 2.2. *A ring R is a ϕ-CR if and only if for every $a, b \in R \setminus Nil(R)$, either $a \mid b$ in R or $b \mid a$ in R. Hence, if R is a ϕ-CR and $x \in T(R) \setminus R$, then $x^{-1} \in R$.*

PROOF. Suppose that R is a ϕ-CR, and let $a, b \in R \setminus Nil(R)$ such that $a \nmid b$ in R. Hence, $b/a \in K \setminus \phi(R)$ by Lemma 2.1. Thus, $a/b \in \phi(R)$. Hence, $b \mid a$ in R by Lemma 2.1. The converse is clear. Now, suppose that R is a ϕ-CR and there is an $x \in T(R) \setminus R$. Then $x = a/b$ for some $a \in R$ and for some $b \in R \setminus Z(R)$ and $b \nmid a$ in R. Hence, $a \mid b$ in R. Since $a \mid b$ and $b \in R \setminus Z(R)$ and $Z(R)$ is divided by Proposition 1.1(6), we conclude that $a \in R \setminus Z(R)$. Thus, $x^{-1} = b/a \in R$. □

Corollary 2.3. (1) *A ϕ-CR is a ϕ-PVR.*
(2) *A ϕ-CR is a divided ring and hence it is quasilocal.*
(3) *A K-CR is a K-PVR.*
(4) *A K-CR is a divided ring and hence it is quasilocal.*
(5) *A homomorphic image of a ϕ-CR is a ϕ-CR.*

PROOF. **(1) and (3).** These are clear by the definitions.
(2) and (4). Since a ϕ-CR (K-CR) is a ϕ-PVR (K-PVR) and a ϕ-PVR (K-PVR) is a divided ring by [8, Proposition 4], the claim follows.
(5). It follows directly from Proposition 2.2. □

In the following result, we construct a ϕ-CR of Krull dimension zero that is not a chained ring.

Proposition 2.4. *Let P be a positive prime number and $n > 1$. Then $A := Z_{P^n}(+)Z_{P^n}$ is a ϕ-CR of Krull dimension zero and A is not a chained ring.*

PROOF. By Proposition 1.2(1) it is clear that $dim(A) = 0$ and $M = Nil(A) = PZ_{P^n}(+)Z_{P^n}$ is the maximal ideal of A. Hence, $M = Nil(A)$ is a divided prime ideal of A. Thus, A is a ϕ-CR. Finally, it is easy to see that neither of the elements $(P, 0)$ and $(0, 1)$ divides the other. Hence, A is not a chained ring. □

To construct a ϕ-CR of Krull dimension ≥ 1 that is not a chained ring, we need the following result

Proposition 2.5. [9, Proposition 2.1]. *Let D be a valuation domain with maximal ideal M and Krull dimension n, say $M = P_n \supset P_{n-1} \supset \ldots \supset P_1 \supset \{0\}$ where the P_i's are the distinct prime ideals of D. Let $i, m, d \geq 1$ such that $1 \leq i \leq m \leq n$. Choose $x \in D$ such that $Rad(x) = P_i$. Let $Q := P_m$ and $J := x^{d+1}D_Q$ and $R := D/J$. Then:*

(1) *J is an ideal of D and $Rad(J) = P_i$.*
(2) *R is a chained ring with maximal ideal M/J and $Z(R) = P_m/J$ and $Nil(R) = P_i/J$. Furthermore, $w := x + J \in Nil(R)$ and $w^d \neq 0$ in R.*
(3) *$dim(R) = n - i$.*
(4) *If $i < m < n$, then $Nil(R)$ is properly contained between $Z(R)$ and M/J.*

In the following result we show that for each $n \geq 1$, there is a ϕ-CR of Krull dimension n that is not a chained ring.

Theorem 2.6. *For each $n \geq 1$, there is a ϕ-CR of Krull dimension n that is not a chained ring.*

PROOF. By Proposition 2.5, there is a chained ring R of Krull dimension n. Let $B = R_{Nil(R)}$ as an R-module and set $A = R(+)B$. It is easy to see that $Nil(A) = Ni(R)(+)B$. Since $Nil(R)$ is a prime ideal of R, $Nil(A)$ is a prime ideal of A by Proposition 1.2(1). We show that $Nil(A)$ is divided. Let $(x, b) \in Nil(A)$ for some $x \in Nil(R)$ and for some $b \in B$, and let $(y, d) \in A \setminus Nil(A)$. Then $y \in R \setminus Nil(R)$ and $d \in B$ and $x = yf$ for some $f \in R$. Hence, $(x, b) = (f, \frac{b-fd}{y})(y, d)$. Thus, $(y, d) \mid (x, b)$ in A. Hence, $Nil(A)$ is a divided prime ideal of A. To see that A is not a chained ring: let $(0, 1)$ and $(x, 0) \in Nil(A)$. It is easy to check that neither one divides the other in A. Hence, A is not a chained ring. Now, we show that A is a ϕ-CR. Let $(a, b), (c, d) \in A \setminus Nil(A)$. Hence, $a, c \in R \setminus Nil(R)$. Thus, either $a \mid c$ in R or $c \mid a$ in R, say $(a, b) \mid (c, d)$. Then $c = az$ for some $z \in R$. Hence, $(c, d) = (z, \frac{d-zb}{a})(a, b)$. Thus, $(a, b) \mid (c, d)$ in A. Hence, A is a ϕ-CR. Now, dim(A) = dim(R) = n by Proposition 1.2(1). □

In view of Proposition 2.5 and the proof of Theorem 2.6, we have the following result.

Corollary 2.7. *Let $d \geq 2$, and $n \geq 2$. Then there is a ϕ-CR A with maximal ideal M and Krull dimension n that is not a chained ring such that $Nil(A)$ is properly contained between $Z(A)$ and M, and $x^d \neq 0$ in A for some $x \in Nil(A)$.*

It is shown in [3, Example 3.16 (c)] that if $(I, \leq)$ is any set which can be realized as the spectrum of some valuation domain and m is the minimum element of I, L is the maximum element of I, and $i \in I$ with $m \leq i \leq L$, then there is a chained ring (R, M) with $Spec(R)$ order-isomorphic to I, where $Nil(R) \leftrightarrow m$, $Z(R) \leftrightarrow i$, and $M \leftrightarrow L$. Hence, in light of this result and the proof of Theorem 2.6, we have the following result.

Corollary 2.8. *Let $d \geq 2$ and $n \geq 2$. Then there is a ϕ-CR A with maximal ideal M that is not a chained ring such that $dim(A)$ is infinite, $w^d \neq 0 (in A)$ for some $w \in Nil(A)$, and $Nil(A)$ is properly contained between $Z(R)$ and M.*

Let R be a ring. We say that B is an overring of R if $R \subset B \subset T(R)$. Also, we say that B is an overring of $\phi(R)$ if $\phi(R) \subset B \subset K$. For the remaining part of this section, we state some results that a ϕ-CR and its "twin" ring (chained ring) enjoy.

Proposition 2.9. *Let R be a ϕ-CR. Then*

(1) *If B is an overring of R, then B is a ϕ-CR and $B = R_P$ for some prime ideal P of R such that $Z(R) \subset P$.*
(2) *If B is an overring of $\phi(R)$, then B is a K-CR and $B = \phi(R)_Q$ for some prime ideal Q of $\phi(R)$.*

PROOF. **(1).** Let B be an overring of R and let $x \in K \setminus \phi(B)$. Then $x \in K \setminus \phi(R)$. Hence, $x^{-1} \in \phi(R)$. Thus, $x^{-1} \in \phi(B)$. Hence, B is a ϕ-CR. Now let M be the maximal ideal of B and $P = M \cap R$. Since B is an overring of R, $Z(R) \subset M$ and therefore $Z(R) \subset P$. Since each $s \in R \setminus P$ is a unit of B, $R_P \subset B$. Now, let $x = a/b \in B$ for some $a \in R$ and for some $b \in R \setminus Z(R)$. If $b \mid a$ in R, then $x \in R_P$. Hence, assume that $b \nmid a$ in R. Then $a \mid b$ in R. Thus, $x = 1/c$ for some $c \in R \setminus Z(R)$. Hence, c is a unit of B. Thus, $c \in R \setminus P$. Hence, $B \subset R_P$. Thus, $B = R_P$.
(2). This is clear by an argument similar to the one just given. □

Proposition 2.10. *Let R be a ϕ-CR. Then R is integrally closed in $T(R)$ and $\phi(R)$ is integrally closed in K.*

PROOF. Let B be the integral closure of R in $T(R)$. Then $B = R_P$ for some prime ideal P of R such that $Z(R) \subset P$ by Proposition 2.9(1). Since $\frac{1}{x}$ is integral over R for some $x \in R \setminus Z(R)$ if and only if x is a unit of R, we see that P must be the maximal ideal of R. Hence, R is integrally closed in $T(R)$. A similar argument shows that $\phi(R)$ is integrally closed in K. □

The following result can be proved by making minor changes in the proof of [14, Theorem 56, page 36].

Proposition 2.11. (1) *Let I be a proper ideal of $\phi(R)$. Then there exists a K-CR V such that $\phi(R) \subset V \subset K$ and $IV \neq V$.*

(2) *If $Nil(R) = Z(R)$ and I is a proper ideal of R, then there exists a ϕ-CR V such that $R \subset V \subset T(R)$ and $IV \neq V$.*

Let I be a proper ideal of R. Then $\phi(I)$ is a proper ideal of $\phi(R)$. Hence, by the above proposition there exists a $K - CR$ such that $\phi(R) \subset V \subset K$ and $\phi(I)V \neq V$.

3. ϕ-CRs and PVRs

Once again, throughout this section R denotes a ring such that $Nil(R)$ is a divided prime ideal of R. The following two lemmas are needed in this section.

Lemma 3.1. (1) *If B, C are ϕ-CRs having the same maximal ideal and $T(B) = T(C)$, then $B = C$.*

(2) *If B, C are overrings of $\phi(R)$ such that B, C are $K - CRs$ having the same maximal ideal, then $B = C$.*

Proof. **(1)**. Suppose that B and C are ϕ-CRs having the same maximal ideal P and $T(B) = T(C)$. We show $B = C$. Suppose there is an $x \in C \setminus B$. Then $x^{-1} \in B$ by Proposition 2.2. Thus, x^{-1} is not a unit in B. Hence, $x^{-1} \in P$ which is impossible, since P is the maximal ideal of C and $x \in C$ and $x^{-1} \in P$. Hence, $C \subset B$. In a similar way, one can show that $B \subset C$. Thus $B = C$.
(2). We just use a similar argument as in (1).

□

Lemma 3.2. *Let B and C be overrings of R. Then $B = C$ if and only if $\phi(B) = \phi(C)$.*

Proof. Suppose that $\phi(B) = \phi(C)$. Let $c \in C$. Then $\phi(c) = \phi(b)$ for some $b \in B$. Since $Nil(R)$ is a divided prime ideal, $Nil(C) = Nil(B) = Nil(R)$. Hence, we may assume that neither c is a nilpotent element of C nor b is a nilpotent element of B. Thus,$\phi(c - b) = 0$. Hence, we have $c - b \in Ker(\phi)$. By [8, Proposition 2(1)], $c - b \in Nil(R) \subset B$. Thus, $c \in B$. Hence, $C \subset B$, and we have B = C by symmetry. □

In the following result, we sharpen [9, Proposition 10]. First, recall that if I is an ideal of R, then $I : I = \{x \in T(R) : xI \subset I\}$, and if J is an ideal of $\phi(R)$, then $J : J = \{x \in K : xJ \subset J\}$.

Proposition 3.3. *Let R be a quasilocal ring with maximal ideal M. Then:*

(1) *Suppose that M contains a nonzerodivisor. Then R is a ϕ-PVR if and only if $M : M$ is a ϕ-CR with maximal ideal M.*

(2) *$\phi(R)$ is a K-PVR with maximal ideal $\phi(M)$ if and only if $\phi(M) : \phi(M)$ is a K-CR with maximal ideal $\phi(M)$.*

PROOF. **(1).** Suppose that R is a ϕ-PVR with maximal ideal M and there is an $s \in M \setminus Z(R)$. Then $M : M$ is a ϕ-PVR with maximal ideal M by [8, Proposition 10(1)]. Hence, we only need to show that if $x, y \in M \setminus Nil(R)$, then either $x \mid y$ in $M : M$ or $y \mid x$ in $M : M$. Suppose that x does not divide y in $M : M$. Then x does not divide y in R. Hence, since R is a ϕ-PVR, $y \mid xs$ in R by Proposition 1.1(6). Thus, $xs = yd$ for some $d \in R$. Suppose that $d \mid s$ in R. Then $d \in R \setminus Z(R)$, since $s \in R \setminus Z(R)$. Hence, $x \mid y$ in R which contradicts our assumption. Thus, d does not divide s (in R). Hence, $s \mid dm$ for each $m \in M$. Thus, $\frac{d}{s}m \in R$ for each $m \in M$. Since d does not divide s in $R, \frac{d}{s}m \in M$ for each $m \in M$. Thus, $\frac{d}{s} \in M : M$. Hence, $x = y\frac{d}{s}$. Thus $y \mid x$ in $M : M$. Thus, $M : M$ is a ϕ-CR. Conversely, suppose that $M : M$ is a ϕ-CR with maximal ideal M. Then $M : M$ is a ϕ-PVR with maximal ideal M by Corollary 2.3(1). Hence, R is a ϕ-PVR by [8, Proposition 10(1)].

(2). Suppose that $\phi(R)$ is a K-PVR with maximal ideal $\phi(M)$. Let $x \in K \setminus \phi(M) : \phi(M)$. Then $x^{-1}\phi(M) \subset \phi(M)$ by [8, Lemma 6]. Thus, $x^{-1} \in \phi(M) : \phi(M)$. Hence, $\phi(M) : \phi(M)$ is a K-CR. Conversely, suppose that $\phi(M) : \phi(M)$ is a K-CR with maximal ideal $\phi(M)$. Then $\phi(M) : \phi(M)$ is a a K-PVR by Corollary 2.3(3). Hence, $\phi(R)$ is a K-PVR by [8, Proposition 10(2)]. □

Corollary 3.4. (1) *Suppose that R is a ϕ-PVR with maximal ideal M containing a nonzerodivisor of R. Then $\phi(M : M) = \phi(M) : \phi(M)(inK)$.*

(2) *A quasilocal ring R with maximal ideal M containing a nonzerodivisor of R is a ϕ-PVR if and only some overring of R is a ϕ-CR with maximal ideal M.*

(3) *Let R be quasilocal with maximal ideal M. Then $\phi(R)$ is a K-PVR if and only if some overring of $\phi(R)$ is a K-CR with maximal ideal ideal $\phi(M)$.*

(4) *If R is quasilocal with maximal ideal M such that $M : M$ is a ϕ-CR, then R is a ϕ-PVR.*

PROOF. **(1).** Since $\phi(M : M)$ is a K-CR with maximal ideal $\phi(M)$ by Proposition 3.3(1) and $\phi(M) : \phi(M)$ is a K-CR with maximal ideal $\phi(M)$ by Proposition 3.3(2), $\phi(M : M) = \phi(M) : \phi(M)$ by Lemma 3.1(2).

(2). Suppose that C is an overring of R with maximal ideal M that is a ϕ-CR. Then $\phi(C) = \phi(M : M)(inK)$ by Lemma 3.1(2). Hence, $C = M : M$ by Lemma 3.2.Thus, the claim is now clear by Proposition 3.3.

(3). The proof is similar to that in (2).
(4). This is clear by the proof of Proposition 3.3(1). □

Lemma 3.5. *Let R be a ϕ-PVR, and P be a prime ideal of R. Then $x^{-1}P \subset P$ for each $x \in T(R) \setminus R$.*

PROOF. Let $x = a/b \in T(R) \setminus R$ for some $a \in R$ and for some $b \in R \setminus Z(R)$. Since $b \not| a$ in R and $Z(R)$ is a divided prime ideal by Proposition 1.1(6), we conclude that $a \in R \setminus Z(R)$. Hence, $x^{-1} = b/a \in T(R)$. Now, let $p \in P$. Then $x(x^{-1}p) = p \in P$. Hence, $\phi(xx^{-1}p) = \phi(x)\phi(x^{-1}p) = \phi(p) \in \phi(P)$. Since $\phi(P)$ is a K-strongly prime ideal of $\phi(R)$ and by Lemma 2.1 $\phi(x) \notin \phi(P)$, we conclude that $\phi(x^{-1}p) \in \phi(P)$. Thus, $\phi(x^{-1}p) = \phi(q)$ for some $q \in P$. Hence, $x^{-1}p - q \in Ker(\phi)$. Since $q \in P$ and $Ker(\phi) \subset Nil(R)$ by [8, Proposition 2(1)] and $Nil(R) \subset P$, we conclude that $x^{-1}p \in P$. □

Proposition 3.6. *Let R be a ϕ-PVR with maximal ideal M, and suppose that C is an overring of R. The following statements are equivalent:*

(1) *C contains an element of the form $1/s$ for some nonzerodivisor s of R.*
(2) *$IC = C$ for some proper ideal I of R.*

PROOF. **(1)⇒ (2).** Let $I = (s)$. Then $IC = C$.
(2)⇒ (1). Suppose that C does not contain an element of the form $1/s$ for some nonzerodivisor $s \in R$ and $IC = C$ for some proper ideal I of R. Let $c \in C \setminus R$. Then $c^{-1} \notin R$. Hence, $cM \subset M$ by Lemma 3.5. In particular, $cI \subset M$. Thus, $IC \subset M$ which is a contradiction. Hence, C contains an element of the form $1/s$ for some nonzerodivisor $s \in R$. □

The proof of the following lemma is very similar to the proof of the above proposition and is therefore omitted.

Lemma 3.7. *Suppose that $\phi(R)$ is a K-PVR and C is an overring of $\phi(R)$. The following statements are equivalent:*

(1) *C contains an element of the form $1/s$ for some nonzerodivisor $s \in \phi(R)$.*
(2) *$IC = C$ for some proper ideal I of $\phi(R)$.*

Proposition 3.8. (1) *Let C be an overring of a ϕ-PVR R such that $IC = C$ for some proper ideal I of R. Then C is a ϕ-CR.*
(2) *Suppose that $\phi(R)$ is a K-PVR, and C is an overring of $\phi(R)$ such that $IC = C$ for some proper ideal I of $\phi(R)$. Then C is a K-CR.*

PROOF. **(1).** By Proposition 3.6, C contains an element of the form $1/s$ for some nonzerodivisor $s \in R$. Now, let $x, y \in C \setminus Nil(C)$ and suppose that x does not

divide y in C. Then, it is easy to check that $y \mid xs$ in C. Hence, $xs = yd$ for some $d \in C$. Thus, $x = y\frac{d}{s}$ and $\frac{d}{s} \in C$ since $1/s \in C$. Hence, $y \mid x$ in C. Thus, C is a ϕ-CR.
(2). In view of Lemma 3.7, we just use a similar argument as in (1). □

Proposition 3.9. (1) *Let P be a nonmaximal prime ideal of a ϕ-PVR R. Then R_P is a ϕ-CR with maximal ideal PR_P.*
(2) *Let P be a nonmaximal prime ideal of a ϕ-PVR R such that $Z(R) \subset P$. Then $R_P = P : P$ is a ϕ-CR with maximal ideal P.*
(3) *Suppose that $\phi(R)$ is a K-PVR, and P is a nonmaximal prime ideal of $\Phi(R)$. Then $P : P = \phi(R)_P$ is a K-CR with maximal ideal P.*

PROOF. **(1).** It is clear that PR_P is the maximal ideal of R_P. Also, since $Nil(R)$ is a divided prime ideal of R, $Nil(R_P) = Nil(R)R_P$ is a divided prime ideal of R_P. Now, let $x, y \in PR_P \setminus Nil(R_P)$. Then $x = a/s$ and $y = b/s$ for some $a, b \in R$ and for some $s \in R \setminus P$. Suppose that x does not divide y in R_P. Then $a \nmid b$ in R. Since P is nonmaximal, there is a nonunit $c \in R \setminus P$. Hence, $b \mid ac$ in R. Thus, $ac = bd$ for some $d \in R$. Hence, $d/c \in R_P$ and $a/s = \frac{b}{s}dc$ in R_P. Thus, $y \mid x$ in R_P.
(2). Since P is a divided prime, $P \subset (x)$ for each $x \in R \setminus P$.Thus, x is a unit in $P : P$ for each $x \in R \setminus P$. Now, let $y \in P : P \setminus R$. Then $y^{-1}P \subset P$ by Lemma 3.5. Hence, $y^{-1} \in P : P$. Thus, y is a unit in $P : P$. Hence, P is the maximal ideal of $P : P$. Since $P : P$ contains an element of the form $1/s$ for some nonunit $s \in R$, $P : P$ is a ϕ-CR by Proposition 3.8. Since R_P is a ϕ-CR with maximal ideal P by (1) and $P : P$ is a ϕ-CR with maximal ideal P, $R_P = P : P$ by Lemma 3.1(1).
(3). We just use a similar argument as in (1) and (2). □

In the next result, we show that a ϕ-PVR is a pullback of a ϕ-CR. If A is a ring, then $Max(A)$ denotes the set of all maximal ideal of A. We recall the following result.

Proposition 3.10. [4, Theorem 3.10] *Let $D \subset E$ be rings. Then $Spec(D) = Spec(E)$ if and only if $Max(E) \subset Max(D)$.*

Proposition 3.11. *Let C be a ϕ-CR with maximal ideal M, $H = C/M$ its residue field, $\alpha : C \longrightarrow H$ be the canonical epimorphism, F a subfield of H, and $R = \alpha^{-1}(F)$. Then the pullback $R = C \times_H F$ is a ϕ-PVR. Moreover, if F is a proper subfield of H, then $R = \alpha^{-1}(F)$ is a ϕ-PVR but not a ϕ-CR.*

PROOF. It is clear that M is a maximal ideal of R. Since $Max(C) \subset Max(R)$, $Spec(R) = Spec(C)$ by Proposition 3.10. Hence, R is quasilocal with maximal ideal M. Since $Nil(C)$ is a divided prime ideal of C, $Nil(R) = Nil(C)$ is a divided ideal of R. Now, let $x, y \in R \setminus Nil(R)$. By Proposition 1.2(6), we need to show that either $x \mid y$ or $y \mid xm$ for each $m \in M$. Since $x, y \in C \setminus Nil(C)$, either $x \mid y$ in C or $y \mid x$ in C. We may assume that $x \mid y$ in C. Now, if $x \mid y$ in R, then we are done. Hence, assume that x does not divide y in R. Since $x \mid y$ in C, $y = xc$ for some c in C. Since $c \notin M$, c is a unit in C. Thus, $yc^{-1} = x$. Now, let $m \in M$. Then $y(c^{-1})m = xm$. Since $c^{-1} \in C$ and M is the maximal ideal of C, $c^{-1}m \in M$. Thus, $y \mid xm$ (in R) for each $m \in M$. Hence, R is a ϕ-PVR. Now, if F is a proper subfield of H, then $R = \alpha^{-1}(F)$ is a proper subring of C. Hence, R is not a ϕ-CR by Lemma 3.1. □

In view of Proposition 3.3(1) and the above proposition, we have the following result.

Corollary 3.12. *Let R be quasilocal ring with maximal ideal M such that M contains a nonzerodivisor of R. Then R is a ϕ-PVR if and only if R is a pullback of a ϕ-CR with maximal ideal M.*

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